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We develop a method of general perturbations mainly applicable to satellite theory. The basic principle is the iterative correction of the frequencies of the angular variables, according to Lindstedt's technique. The solutions are developed as power series of a small parameter and are closely related to those given by von Zeipel's method. They differ in that they are obtained by direct integration of the differential equations of motion so that they can also be developed for nonconservative systems. For a system with three degrees of freedom, we give the differential equations for any order of approximation and develop explicit relations up to the third order in the small parameters.

1. INTRODUCTION

It is well known that the classical method of Poincaré and von Zeipel applies to conservative Hamiltonian systems. Moreover, the construction of a generating function for high-order perturbations may become very involved, especially when we deal with short-period terms (Kozai 1962). The construction of an algorithm suitable for automatic symbolic processing is not a trivial task and has been developed only in certain problems and to a maximum of the second order. Another nontrivial task is the relation between initial conditions and the element constants as defined by Brouwer (1959) and Garfinkel (1959). In his doctoral dissertation, the author (1965) presented a sketch of a direct evaluation of von Zeipel's series based on the work of Poincaré (1893). Since that time, the need for a high-order solution and the remarkable progress in automatic processing of algebraic symbols have shown that such a method would have great advantages over the classical ones. We shall deal initially with one-dimensional systems, in order to explain better the process of solution. We then develop a theory for three-dimensional systems. The generalization to a system with more than three degrees of freedom is straightforward. The study of a system with one degree of freedom has, in addition to other purposes, that of separating the problem of convergence of series in a small parameter from that introduced by small divisors. The applicability to automatic processing derives mostly from the fact that we can obtain the equations that produce the terms corresponding to the *n*th order of approximation. The operations needed for purely analytic development are multiplication of Fourier's series and integration of such series in time. For semianalytic development, we need also the Fourier analysis of the disturbing function and its derivatives. All these operations can be satisfactorily developed by any high-speed electronic calculator. It should be noted that the general equations for von Zeipel's method are much more cumbersome and, in general, represent systems of partial differential equations (Giacaglia 1964, 1965).

2. ONE DEGREE OF FREEDOM

Although it is not strictly necessary, we will consider a Hamiltonian system, and call ℓ the coordinate (angular variable) and L the generalized associated momentum (action variable). The negative of the Hamiltonian is given by

$$F = F(L, \ell ; \epsilon) = F_0(L) + \epsilon \mathcal{F}(L, \ell) \quad , \tag{1}$$

where ϵ is a constant dimensionless parameter defined in (0, 1), but usually small compared with unity.

We suppose F to be analytic in the neighborhood of L_0 for $|L - L_0| < R$ (where R is a given number), to be capable of being developed in a convergent Fourier series in ℓ, and to be periodic of period 2π in this variable. Furthermore, we suppose as usual that

$$\int_0^{2\pi} \mathcal{F}(L, \ell) \, d\ell = 0 \quad ,$$

which can always be achieved by assimilation of the "secular" part of $\mathcal{F}$ into F_0, which in general might depend on ϵ. This dependence is disregarded altogether in the process, with no confusion in the final outcome.

The differential equations pertinent to $F(L, \ell)$ are

$$\dot{L} = \frac{\partial F}{\partial \ell} \quad , \quad \dot{\ell} = - \frac{\partial F}{\partial L} \quad , \tag{2}$$

or, according to Eq. (1),

$$\dot{L} = \epsilon \frac{\partial \mathcal{F}}{\partial \ell} \quad , \quad \dot{\ell} = N(L) - \epsilon \frac{\partial \mathcal{F}}{\partial L} \quad , \tag{3}$$

where

$$N(L) = - \frac{\partial F_0}{\partial L} \quad . \tag{4}$$

We consider the new variables

$$x = L - L_0 \quad , \quad |x| < R$$

$$y = \ell - \omega = \ell - (\nu t + \beta) \quad , \tag{5}$$

where $\nu = \nu(L_0;\epsilon)$ is an unknown function of a constant L_0, which is related to the initial conditions. Moreover, the function $\nu(L_0;\epsilon)$ is supposed to be analytic in the neighborhood of $\epsilon = 0$; that is, it can be developed in convergent power series

$$\nu = \nu_0 + \epsilon \nu_1 + \epsilon^2 \nu_2 + \ldots \quad , \tag{6}$$

where

$$\nu_k = \nu_k(L_0) \quad .$$

By hypothesis, we can write

$$\mathcal{F} = \sum_{k \neq 0} [A_k(L) \cos k\ell + B_k(L) \sin k\ell] \quad ,$$

where the integer k takes all values from $-\infty$ to $+\infty$, and A_k, B_k are analytic functions of L.

In terms of the new variables x and y, Eqs. (3) can be written

$$\dot{x} = \epsilon \frac{\partial \mathcal{F}}{\partial \ell}$$

$$\dot{y} = N(L) - \nu(L_0;\epsilon) - \epsilon \frac{\partial \mathcal{F}}{\partial L} \quad . \tag{7}$$

We have

$$\frac{\partial \mathcal{F}}{\partial \ell} = \sum_{k \neq 0} k(-A_k \sin k\ell + B_k \cos k\ell)$$

$$\frac{\partial \mathcal{F}}{\partial L} = \sum_{k \neq 0} (A_k^{(1)} \cos k\ell + B_k^{(1)} \sin k\ell) \quad , \tag{8}$$

where $A_k^{(n)} = d^n A_k / dL^n$, and similar notation is applicable to other functions. Hence, Eqs. (7) can be written as

$$\dot{x} = \epsilon \sum_{k \neq 0} k[-A_k(L_0 + x) \sin k(\omega + y) + B_k(L_0 + x) \cos k(\omega + y)]$$

$$\dot{y} = N(L_0 + x) - \nu(L_0;\epsilon)$$

$$- \epsilon \sum_{k \neq 0} [A_k^{(1)}(L_0 + x) \cos k(\omega + y) + B_k^{(1)}(L_0 + x) \sin k(\omega + y)] \quad . \tag{9}$$

The next step is the development of the right-hand members of Eqs. (9) in double Taylor's series, in the neighborhood of (L_0, ω). We have

$$f(L_0 + x, \omega + y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x \frac{\partial}{\partial L_0} + y \frac{\partial}{\partial \omega} \right)^n f(L_0, \omega)$$

and

$$\frac{d^n}{d\ell^n} \begin{matrix} \sin \\ \cos \end{matrix} (k\ell) = (-1)^n k^n \begin{matrix} \sin \\ \cos \end{matrix} (k\ell - n\frac{\pi}{2}) \quad .$$

Therefore, Eqs. (9) become

$$\dot{x} = \epsilon \sum_{n=0}^{\infty} \sum_{m=0}^{n} X_{n-m,m}(\omega, L_0) x^{n-m} y^m$$

$$\dot{y} = -\nu(L_0; \epsilon) + \sum_{n=0}^{\infty} a_n(L_0) x^n - \epsilon \sum_{n=0}^{\infty} \sum_{m=0}^{n} Y_{n-m,m}(\omega, L_0) x^{n-m} y^m \quad , \tag{10}$$

where

$$a_n(L_0) = \frac{1}{n!} N^{(n)}(L_0) = -\frac{1}{n!} F_0^{(n+1)}(L_0) \tag{11}$$

and $X_{n-m,m}$ and $Y_{n-m,m}$ are defined by the Fourier series

$$X_{n-m,m} = \sum_{k\neq 0} \frac{(-1)^m k^{m+1}}{n!} \binom{n}{m}$$

$$\times [-A_k^{(n-m)}(L_0) \sin (k\omega - m\frac{\pi}{2}) + B_k^{(n-m)}(L_0) \cos (k\omega - m\frac{\pi}{2})]$$

$$Y_{n-m,m} = \sum_{k\neq 0} \frac{(-1)^m k^m}{n!} \binom{n}{m}$$

$$\times [A_k^{(n-m+1)}(L_0) \cos (k\omega - m\frac{\pi}{2}) + B_k^{(n-m+1)}(L_0) \sin (k\omega - m\frac{\pi}{2})] \quad . \tag{12}$$

The right-hand members of Eqs. (10) converge provided

$$|x| = |L - L_0| < R < M$$

$$|y| = |\ell - \omega| < M$$

with a properly chosen positive quantity M. These requirements will be met if x and y can be defined as purely periodic functions of ω, with bounded coefficients. The method consists of constructing such functions, by successive approximations, in the form of Fourier series. Systems of type (10) have been considered by McMillan (1920), who studied the convergence of solutions x, y that are power series in ϵ. Because McMillan did not require x and y to be periodic, he could only prove convergence for a finite interval of time. Since we will show that x and y can be constructed as periodic functions, it is possible to prove the convergence of their series representation, provided ϵ is small enough and some values of L_0 are excluded. The proof follows the lines described by Moser (1967). We shall limit ourselves to showing that it is possible to construct the formal series

$$\nu = \nu_0 + \epsilon\nu_1 + \epsilon^2\nu_2 + \ldots \quad , \quad \nu_k = \nu_k(L_0) \quad ,$$

$$x = \epsilon x_1 + \epsilon^2 x_2 + \ldots \quad , \quad x_k = x_k(L_0, \omega) \quad ,$$

$$y = \epsilon y_1 + \epsilon^2 y_2 + \ldots \quad , \quad y_k = y_k(L_0, \omega) \quad .$$

If these series are substituted into Eqs. (10), we obtain

$$\sum_{p=1}^{\infty} \epsilon^p \dot{x}_p = \epsilon \sum_{n=0}^{\infty} \sum_{m=0}^{n} X_{n-m,m}\, \epsilon^n$$

$$\times \sum_{s_1=1}^{\infty} \cdots \sum_{s_{n-m}=1}^{\infty} x_{s_1} \cdots x_{s_{n-m}}\, \epsilon^{s_1+\ldots+s_{n-m}-n+m}$$

$$\times \sum_{r_1=1}^{\infty} \cdots \sum_{r_m=1}^{\infty} y_{r_1} \cdots y_{r_m} \epsilon^{r_1+\cdots+r_m-m} \tag{12}$$

$$\sum_{p=1}^{\infty} \epsilon^p \dot{y}_p = -\sum_{p=1}^{\infty} \epsilon^p \nu_p + \sum_{n=1}^{\infty} a_n \epsilon^n$$

$$\times \sum_{s_1=1}^{\infty} \cdots \sum_{s_n=1}^{\infty} x_{s_1} \cdots x_{s_n} \epsilon^{s_1+\cdots+s_n-n} - [\dot{x}(X \rightarrow Y)] \quad , \tag{13}$$

where we have set $\nu_0 = a_0 = - F_0^{(1)}(L_0)$ and the expression $[\dot{x}(X \rightarrow Y)]$ indicates the right-hand member of Eq. (12) where the Y's are substituted for the X's.

By equating coefficients of the same powers of ϵ in both sides of Eqs. (12) and (13), we obtain the differential equations for the unknowns x_k, y_k, ν_k $(k = 1, 2, 3, \ldots)$. It follows immediately that

$$\dot{x}_p = \sum_{n=0}^{p-1} \sum_{m=0}^{n} X_{n-m,m} \sum_{s_1+\cdots+s_{n-m}+r_1+\cdots+r_m=p-1} x_{s_1} \cdots x_{s_{n-m}} y_{r_1} \cdots y_{r_m} \tag{14}$$

$$\dot{y}_p = -\nu_p + \sum_{n=1}^{p} a_n \sum_{s_1+\cdots+s_n=p} x_{s_1} \cdots x_{s_n} - [\dot{x}_p(X \rightarrow Y)] \quad , \tag{15}$$

where $s_j \geq 1$, $r_k \geq 1$ $(j = 1, \ldots, n-m;\ k = 1, \ldots, m)$ and for $p > 1$, $n \geq 1$.

In particular, we have

(1) $p = 1$:

$$\dot{x}_1 = X_{0,0}$$

$$\dot{y}_1 = -\nu_1 + a_1 x_1 - Y_{0,0} \quad .$$

Since $X_{0,0}$ and $Y_{0,0}$ are purely periodic, the choice for ν_1 is $\nu_1 = 0$. The functions

$$x_1 = \frac{1}{\nu} \int X_{0,0} \, d\omega$$

$$y_1 = \frac{1}{\nu} \int (a_1 x_1 - Y_{0,0}) \, d\omega$$

will then be purely periodic, with no constant term. In fact,

$$x_1 = \frac{1}{\nu} \int \frac{\partial \mathcal{F}}{\partial \omega} (L_0, \omega) \, d\omega = \frac{1}{\nu} \mathcal{F}(L_0, \omega) \tag{16}$$

and

$$y_1 = - \frac{\partial^2 F_0(L_0)}{\partial L_0^2} \frac{1}{\nu^2} \int \mathcal{F}(L_0, \omega) \, d\omega - \frac{1}{\nu} \int \frac{\partial \mathcal{F}(L_0, \omega)}{\partial L_0} \, d\omega$$

$$= - \frac{1}{\nu^2} \frac{\partial^2 F_0(L_0)}{\partial L_0^2} \int \mathcal{F}(L_0, \omega) \, d\omega - \frac{1}{\nu} \frac{\partial}{\partial L_0} \int \mathcal{F}(L_0, \omega) \, d\omega$$

or

$$y_1 = - \left(\frac{1}{\nu^2} F_0'' + \frac{1}{\nu} \frac{\partial}{\partial L} \right) \int \mathcal{F}(L_0, \omega) \, d\omega \quad , \tag{17}$$

where it is to be noted that to a first-order approximation $\nu = \nu_0 + \epsilon \nu_1 = \nu_0 = - F_0'$.

(2) $p = 2$:

$$\dot{x}_2 = X_{1,0}x_1 + X_{0,1}y_1$$

$$\dot{y}_2 = -\nu_2 + a_1x_2 + a_2x_1^2 - Y_{1,0}x_1 - Y_{0,1}y_1 \quad .$$

The right-hand member of $\dot{x}_2$ cannot contain terms independent of ω. In order to see this, we first note that

$$X_{1,0} = \frac{\partial^2 \mathcal{F}(L_0, \omega)}{\partial\omega \; \partial L_0}$$

$$X_{0,1} = \frac{\partial^2 \mathcal{F}(L_0, \omega)}{\partial\omega^2}$$

and consider a particular argument $\theta = \bar{k}\omega$ in $\mathcal{F}(L_0, \omega)$. Constant terms can only arise from combinations $\cos^2 \theta$ or $\sin^2 \theta$. We have

$$\mathcal{F}_\theta = a \cos \theta + b \sin \theta$$

and therefore

$$x_1 = \frac{1}{\nu}(a \cos \theta + b \sin \theta)$$

$$y_1 = -\left(\frac{F_0''}{\nu^2} + \frac{1}{\nu} \frac{\partial}{\partial L}\right)\frac{1}{\bar{k}}(a \sin \theta - b \cos \theta)$$

$$X_{1,0} = \bar{k}(-a' \sin \theta + b' \cos \theta)$$

$$X_{0,1} = -\bar{k}^2(a \cos \theta + b \sin \theta) \quad .$$

Therefore,

$$\dot{x}_2(\theta) = \frac{\bar{k}}{\nu}(a \cos\theta + b \sin\theta)(- a' \sin\theta + b' \cos\theta)$$

$$+ \frac{\bar{k}\, F''_0}{\nu^2}(a \sin\theta - b \cos\theta)(a \cos\theta + b \sin\theta)$$

$$+ \frac{\bar{k}}{\nu}(a' \sin\theta - b' \cos\theta)(a \cos\theta + b \sin\theta)$$

$$= \frac{\bar{k}\, F''_0}{\nu^2}[(a^2 - b^2) \sin\theta \cos\theta - ab \cos 2\theta] \quad ,$$

which proves our statement.

It follows that x_2, defined by

$$x_2 = \int (X_{1,0} x_1 + X_{0,1} y_1)\, d\omega \quad ,$$

is free from secular terms. We define

$$2\pi\nu_2 = \int_0^{2\pi} (a_2 x_1^2 - Y_{1,0} x_1 - Y_{0,1} y_1)\, d\omega \quad ,$$

and it is easily seen that each part of the integrand will contribute to ν_2. With such a definition of ν_2, the function

$$y_2 = \frac{1}{\nu} \int_\epsilon (- \nu_2 + a_1 x_2 + a_2 x_1^2 - Y_{1,0} x_1 - Y_{0,1} y_1)\, d\omega$$

will be free from secular terms. We note that, for a non-Hamiltonian system, the right-hand member of x_n will in general contain constant terms producing secular perturbations, that is, terms linear in time (or ω). In such cases, except for very special situations, the solution will converge only for a limited interval of time.

(3) $p = 3$:

$$\dot{x}_3 = X_{1,0}x_2 + X_{0,1}y_2 + X_{1,1}x_1y_1 + X_{2,0}x_1^2 + X_{0,2}y_1^2$$

and

$$\dot{y}_3 = -\nu_3 + a_1x_3 + 2a_2x_1x_2 + a_3x_1^3 - [\dot{x}_3(X \to Y)] \quad .$$

The task of showing that the right-hand member of $\dot{x}_3$ contains no constant terms follows the same reasoning we applied to the second-order solution. For the general proof we refer to the literature (Giacaglia 1967).

From Eqs. (14) and (15) we see that in general we can write

$$\dot{x}_n = X_n(L_0, \omega;\ x_1, x_2, \ldots, x_{n-1};\ y_1, y_2, \ldots, y_{n-1}) \tag{18}$$

$$\dot{y}_n = -\nu_n + a_1x_n + Y_n(L_0, \omega;\ x_1, x_2, \ldots, x_{n-1};\ y_1, y_2, \ldots, y_{n-1}) \quad . \tag{19}$$

Equation (18) gives x_n if x_k, y_k ($k = 1, 2, \ldots, n-1$) are known as functions of L_0 and ω. Moreover, if we know these functions, ν_n is determined as

$$2\pi\nu_n = \int_0^{2\pi} Y_n \, d\omega \quad , \tag{20}$$

and we then obtain y_n. This completes the description of the solution by recurrence. The series obtained, truncated at the nth stage, are

$$\nu_{(n+1)} = \nu_0 + \epsilon^2\nu_2 + \epsilon^3\nu_3 + \ldots + \epsilon^{n+1}\nu_{n+1}$$

$$x_{(n)} = \epsilon x_1 + \epsilon^2 x_2 + \ldots + \epsilon^n x_n$$

$$y_{(n)} = \epsilon y_1 + \epsilon^2 y_2 + \ldots + \epsilon^n y_n \quad ,$$

where the function ν_{n+1} is included because, as is evident from Eqs. (20) and (19), it does not require that we know $x_{(n+1)}$ and $y_{(n+1)}$. To this order of approximation, the final solution is given by

$$L_{(n)} = L_0 + x_{(n)}$$

$$\ell_{(n)} = \nu_{(n+1)}t + \beta + y_{(n)} \quad , \tag{21}$$

and the frequency ν is known to an approximation one order higher. Equations (21) are implicit relations between the initial conditions $L(0)$, $\ell(0)$ and the constants of integration L_0, β. More will be said about this in the next section.

3. THREE DEGREES OF FREEDOM

In the subsequent development we will deal with a more general Hamiltonian of the form

$$F = F_0(L_1, L_2, L_3) + \epsilon\, \mathcal{F}(L_1, L_2, L_3; \ell_1, \ell_2, \ell_3; \epsilon)$$

and assume that $\mathcal{F}$ can be developed in convergent power series of ϵ. Thus,

$$F = F_0(\underset{\sim}{L}) + \epsilon F_1(\underset{\sim}{L}; \underset{\sim}{\ell}) + \epsilon^2 F_2(\underset{\sim}{L}; \underset{\sim}{\ell}) + \ldots \quad , \tag{22}$$

where $\underset{\sim}{L}$ and $\underset{\sim}{\ell}$ indicate the triplets (L_1, L_2, L_3) and (ℓ_1, ℓ_2, ℓ_3), respectively. The process to be developed is a generalization of what we did for one-dimensional systems. Each $F_p(\underset{\sim}{L}, \underset{\sim}{\ell})$ can be written as

$$F_p(\underset{\sim}{L}, \underset{\sim}{\ell}) = \sum_{k_1, k_2, k_3} [A(p;k_1, k_2, k_3;\underset{\sim}{L}) \cos (k_1\ell_1 + k_2\ell_2 + k_3\ell_3)$$

$$+ B(p;k_1, k_2, k_3;\underset{\sim}{L}) \sin (k_1\ell_1 + k_2\ell_2 + k_3\ell_3)] \quad , \tag{23}$$

where $p = 1, 2, \ldots,$ and $|k_1| + |k_2| + |k_3| \neq 0$. To shorten the notation, we will write (23) as

$$F_p(\underset{\sim}{L}, \underset{\sim}{\ell}) = \sum_{\overline{k}} [A(p;\underset{\sim}{k}) \cos \underset{\sim}{k} \cdot \underset{\sim}{\ell} + B(p;\underset{\sim}{k}) \sin \underset{\sim}{k} \cdot \underset{\sim}{\ell}] \quad ,$$

and the dependence of A and B upon $\underset{\sim}{L}$ is implicitly admitted. In the following developments, $\underset{\sim}{L}^0 \equiv (L_1^0, L_2^0, L_3^0)$, a triplet of constants of integration, will be substituted for $\underset{\sim}{L}$.

The differential equations to be integrated are

$$\dot{L}_i = \epsilon \frac{\partial \mathcal{F}}{\partial \ell_i} = \epsilon \frac{\partial F_1}{\partial \ell_i} + \epsilon^2 \frac{\partial F_2}{\partial \ell_i} + \ldots = \sum_{p=1}^{\infty} \epsilon^p \frac{\partial F_p}{\partial \ell_i} \tag{24}$$

and

$$\dot{\ell}_i = - \frac{\partial F_0}{\partial L_i} - \sum_{p=1}^{\infty} \epsilon^p \frac{\partial F_p}{\partial L_i} \quad , \tag{25}$$

for $i = 1, 2, 3$.

The transformation

$$x_i = L_i - L_i^0$$

$$y_i = \ell_i - \omega_i = \ell_i - (\nu_i t + \beta_i) \quad , \quad i = 1, 2, 3$$

gives

$$\dot{x}_i = \sum_{p=1}^{\infty} \epsilon^p \frac{\partial F_p}{\partial \ell_i}$$

$$\dot{y}_i = - \nu_i - \frac{\partial F_0}{\partial L_i} - \sum_{p=1}^{\infty} \epsilon^p \frac{\partial F_p}{\partial L_i} \quad . \tag{26}$$

The functions

$$- \frac{\partial F_0}{\partial L_i} = N_i(\underset{\sim}{L})$$

$$\frac{\partial F_p}{\partial \ell_i} = \sum_{\underset{\sim}{k}} k_i [- A(p;\underset{\sim}{k}) \sin \underset{\sim}{k} \cdot \underset{\sim}{\ell} + B(p;\underset{\sim}{k}) \cos \underset{\sim}{k} \cdot \underset{\sim}{\ell}]$$

$$\frac{\partial F_p}{\partial L_i} = \sum_{\underset{\sim}{k}} \frac{\partial}{\partial L_i} [A(p;\underset{\sim}{k}) \cos \underset{\sim}{k} \cdot \underset{\sim}{\ell} + B(p;\underset{\sim}{k}) \sin \underset{\sim}{k} \cdot \underset{\sim}{\ell}] \quad ,$$

$$i = 1, 2, 3; \; p = 1, 2, \ldots$$

are to be developed in Taylor's series in the neighborhood of $(\underset{\sim}{L}^0, \underset{\sim}{\omega})$. We have the following general relation:

$$f(\underset{\sim}{x}_0 + \underset{\sim}{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x_1 \frac{\partial}{\partial x_{01}} + x_2 \frac{\partial}{\partial x_{02}} + \ldots \right)^n f(\underset{\sim}{x}_0) \quad ,$$

which, applied to the above functions, gives

$$N_i(\underset{\sim}{L}^0 + \underset{\sim}{x}) = \sum_{n=0}^{\infty} \sum_{i_3=0}^{n} \sum_{i_2=0}^{n-i_3} Z^{(i)}_{i_1, i_2, i_3} \, x_1^{i_1} x_2^{i_2} x_3^{i_3} \quad , \tag{27}$$

where $i_1 = n - i_2 - i_3$, and

$$Z^{(i)}_{i_1, i_2, i_3} = \frac{1}{n!} \binom{n}{i_3} \binom{n-i_3}{i_2} \frac{\partial^n N_i(\underset{\sim}{L}^0)}{\partial {L_1^0}^{i_1} \, \partial {L_2^0}^{i_2} \, \partial {L_3^0}^{i_3}} \quad .$$

From now on we will drop the superscript 0; that is, we will write $\underset{\sim}{L}$ instead of $\underset{\sim}{L}^0$. Therefore,

$$Z^{(i)}_{i_1, i_2, i_3} = - \frac{1}{n!} \binom{n}{i_3} \binom{n-i_3}{i_2} \frac{\partial^{n+1} F_0}{\partial L_i \, \partial L_1^{i_1} \, \partial L_2^{i_2} \, \partial L_3^{i_3}} \quad , \qquad (28)$$

where $i_1 = n - i_3 - i_2$.

Moreover,

$$\frac{\partial F_p}{\partial \ell_i} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{j_3=0}^{n-m} \sum_{j_2=0}^{n-m-j_3} \sum_{\ell_3=0}^{m} \sum_{\ell_2=0}^{m-\ell_3} X^{(i)}_{p;j_i, j_2, j_3;\ell_1, \ell_2, \ell_3}$$

$$\times \; x_1^{j_1} \, x_2^{j_2} \, x_3^{j_3} \, y_1^{\ell_1} \, y_2^{\ell_2} \, y_3^{\ell_3} \quad , \qquad (29)$$

where

$$j_1 = n - m - j_2 - j_3 \quad ,$$

$$\ell_1 = m - \ell_2 - \ell_3 \quad ,$$

and

$$X^{(i)}_{p;j_1,j_2,j_3;\ell_1,\ell_2,\ell_3} = \sum_{\underset{\sim}{k}} \frac{k_i}{n!} \binom{n}{m} \binom{n-m}{j_3} \binom{n-m-j_3}{j_2} \binom{m}{\ell_3} \binom{m-\ell_3}{\ell_2} (-1)^m$$

$$\times\ k_1^{m-\ell_2-\ell_3}\ k_2^{\ell_2}\ k_3^{\ell_3}$$

$$\times \left[- \frac{\partial^{n-m} A(p;\underset{\sim}{k})}{\partial L_1^{j_1}\ \partial L_2^{j_2}\ \partial L_3^{j_3}} \sin (\underset{\sim}{k} \cdot \underset{\sim}{\omega} - m \frac{\pi}{2}) \right.$$

$$\left. + \frac{\partial^{n-m} B(p;\underset{\sim}{k})}{\partial L_1^{j_1}\ \partial L_2^{j_2}\ \partial L_3^{j_3}} \cos (\underset{\sim}{k} \cdot \underset{\sim}{\omega} - m \frac{\pi}{2}) \right] . \tag{30}$$

Finally,

$$\frac{\partial F_p}{\partial L_i} = \left[\frac{\partial F_p}{\partial \ell_i} (X \rightarrow Y) \right] , \tag{31}$$

where

$$Y^{(i)}_{p;j_1,j_2,j_3;\ell_1,\ell_2,\ell_3} = \sum_{\underset{\sim}{k}} \frac{1}{n!} \binom{n}{m} \binom{n-m}{j_3} \binom{n-m-j_3}{j_2} \binom{m}{\ell_3} \binom{m-\ell_3}{\ell_2} (-1)^m$$

$$\times\ k_1^{m-\ell_2-\ell_3}\ k_2^{\ell_2}\ k_3^{\ell_3}$$

$$\times \left[\frac{\partial^{n-m+1} A(p;\underset{\sim}{k})}{\partial L_i\ \partial L_1^{j_1}\ \partial L_2^{j_2}\ \partial L_3^{j_3}} \cos (\underset{\sim}{k} \cdot \underset{\sim}{\omega} - m \frac{\pi}{2}) \right.$$

$$+ \frac{\partial^{n-m+1} B(p;\underset{\sim}{k})}{\partial L_i \, \partial L_1^{j_1} \, \partial L_2^{j_2} \, \partial L_3^{j_3}} \sin\left(\underset{\sim}{k} \cdot \underset{\sim}{\omega} - m\frac{\pi}{2}\right)\Bigg] \tag{32}$$

and (j_1, ℓ_1) are defined as before.

We now introduce the series

$$x_i = \epsilon \sum_{s=1}^{\infty} \epsilon^{s-1} x_i^{(s)}$$

$$y_i = \epsilon \sum_{r=1}^{\infty} \epsilon^{r-1} y_i^{(r)} \quad , \tag{33}$$

which are to be substituted into the above developments and then into differential Eq. (26). The result is

$$\dot{x}_i = \sum_{p=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{j_3=0}^{n-m} \sum_{j_2=0}^{n-m-j_3} \sum_{\ell_3=0}^{m} \sum_{\ell_2=0}^{m-\ell_3} X^{(i)}_{p;j_1,j_2,j_3;\ell_1,\ell_2,\ell_3} \epsilon^{p+n}$$

$$\times \sum_{\alpha_1=1}^{\infty} \cdots \sum_{\alpha_{j_1}=1}^{\infty} \sum_{\beta_1=1}^{\infty} \cdots \sum_{\beta_{j_2}=1}^{\infty} \sum_{\gamma_1=1}^{\infty} \cdots \sum_{\gamma_{j_3}=1}^{\infty} \sum_{\lambda_1=1}^{\infty} \cdots$$

$$\sum_{\lambda_{\ell_1}=1}^{\infty} \sum_{\mu_1=1}^{\infty} \cdots \sum_{\mu_{\ell_2}=1}^{\infty} \sum_{\nu_1=1}^{\infty} \cdots \sum_{\nu_{\ell_3}=1}^{\infty}$$

$$\epsilon^{\alpha_1+\dots+\alpha_{j_1}+\beta_1+\dots+\beta_{j_2}+\gamma_1+\dots+\gamma_{j_3}}$$

$$\times\ \epsilon^{\lambda_1+\dots+\lambda_{\ell_1}+\mu_1+\dots+\mu_{\ell_2}+\nu_1+\dots+\nu_{\ell_3}-n}$$

$$\times\ x_1^{(\alpha_1)} \dots x_1^{(\alpha_{j_1})} x_2^{(\beta_1)} \dots x_2^{(\beta_{j_2})} x_3^{(\gamma_1)} \dots x_3^{(\gamma_{j_3})}$$

$$\times\ y_1^{(\lambda_1)} \dots y_1^{(\lambda_{\ell_1})} y_2^{(\mu_1)} \dots y_2^{(\mu_{\ell_2})} y_3^{(\nu_1)} \dots y_3^{(\nu_{\ell_3})} \quad , \tag{34}$$

and

$$\dot{y}_i = -\epsilon \sum_{k=1}^{\infty} \epsilon^{k-1}\, \nu_i^{(k)} + \sum_{n=0}^{\infty} \sum_{i_3=0}^{n} \sum_{i_2=0}^{n-i_3} Z^{(i)}_{i_1, i_2, i_3}\, \epsilon^{n}$$

$$\times \sum_{\alpha_1=1}^{\infty} \dots \sum_{\alpha_{i_1}=1}^{\infty} \sum_{\beta_1=1}^{\infty} \dots \sum_{\beta_{i_2}=1}^{\infty} \sum_{\gamma_1=1}^{\infty} \dots \sum_{\gamma_{i_3}=1}^{\infty}$$

$$\times\ \epsilon^{\alpha_1+\dots+\alpha_{i_1}+\beta_1+\dots+\beta_{i_2}+\gamma_1+\dots+\gamma_{i_3}-n}$$

$$\times\ x_1^{(\alpha_1)} \dots x_1^{(\alpha_{i_1})} x_2^{(\beta_1)} \dots x_2^{(\beta_{i_2})} x_3^{(\gamma_1)} \dots x_3^{(\gamma_{i_3})}$$

$$-\ [\dot{x}_i(X \rightarrow Y)] \quad , \tag{35}$$

where, as already defined,

$$j_1 = n - m - j_2 - j_3 \quad ,$$

$$\ell_1 = m - \ell_2 - \ell_3 \quad ,$$

$$i_1 = n - i_2 - i_3 \quad .$$

Equations (34) and (35), despite the cumbersome aspect due to their complete generality, give simple relations for the equations that define $x_i^{(k)}$, $y_i^{(k)}$, $k = 1, 2, \ldots$, by recurrence. These are readily found to be

$$\dot{x}_i^{(N)} = \sum_{p=1}^{N} \sum_{n=q}^{N-p} \sum_{m=0}^{n} \sum_{j_3=0}^{n-m} \sum_{j_2=0}^{n-m-j_2} \sum_{\ell_3=0}^{m} \sum_{\ell_2=0}^{m-\ell_3} X^{(i)}_{p;j_1,j_2,j_3;\ell_1,\ell_2,\ell_3}$$

$$\times \sum_{\alpha_1+\ldots+\alpha_{j_1}+\beta_1+\ldots+\beta_{j_2}+\gamma_1+\ldots+\gamma_{j_3}+\lambda_1+\ldots+\lambda_{\ell_1}+\mu_1+\ldots+\mu_{\ell_2}+\nu_1+\ldots+\nu_{\ell_3}=N-p}$$

$$x_1^{(\alpha_1)} \ldots x_1^{(\alpha_{j_1})} x_2^{(\beta_1)} \ldots x_2^{(\beta_{j_2})} x_3^{(\gamma_1)} \ldots x_3^{(\gamma_{j_3})} y_1^{(\lambda_1)} \ldots y_1^{(\lambda_{\ell_1})} y_2^{(\mu_1)} \ldots$$

$$\times y_2^{(\mu_{\ell_2})} y_3^{(\nu_1)} \ldots y_3^{(\nu_{\ell_3})} \quad , \tag{36}$$

where $q = 1 - \delta_{0,N-p}$, and

$$\dot{y}_i^{(N)} \doteq -\nu_i^{(N)} + \sum_{n=1}^{N} \sum_{i_3=0}^{n} \sum_{i_2=0}^{n-i_3} Z^{(i)}_{i_1,i_2,i_3} \sum_{\alpha_1+\ldots+\alpha_{i_1}+\beta_1+\ldots+\beta_{i_2}+\gamma_1+\ldots+\gamma_{i_3}=N}$$

$$x_1^{(\alpha_1)} \ldots x_1^{(\alpha_{i_1})} x_2^{(\beta_1)} \ldots x_2^{(\beta_{i_2})} x_3^{(\gamma_1)} \ldots x_3^{(\gamma_{i_3})} - [\dot{x}_i^{(N)}(X \rightarrow Y)] \quad , \tag{37}$$

where we have already defined

$$\nu_i^{(0)} = Z^{(i)}_{0,0,0} = N_i(\underset{\sim}{L}) = -\frac{\partial F_0}{\partial L_i} \quad , \tag{38}$$

and

$$i_1 = n - i_2 - i_3 \quad ,$$

$$j_1 = n - m - j_2 - j_3 \quad ,$$

$$\ell_1 = m - \ell_2 - \ell_3 \quad .$$

Next, we give Eqs. (36) and (37) for N = 1, 2, 3 (third-order solution).

<u>First Order</u>.

$$\dot{x}_i^{(1)} = X^{(i)}_{1;0,0,0;0,0,0} \quad ,$$

$$\dot{y}_i^{(1)} = -\nu_i^{(1)} + Z^{(i)}_{1,0,0}\, x_1^{(1)} + Z^{(i)}_{0,1,0}\, x_2^{(1)} + Z^{(i)}_{0,0,1}\, x_3^{(1)} - Y^{(i)}_{1;0,0,0;0,0,0} \quad ,$$

so that $\nu_i^{(1)} = 0$ $(i = 1, 2, 3)$,

$$\dot{x}_i^{(1)} = \frac{\partial F_1}{\partial \omega_i} \quad , \quad x_i^{(1)} = \int \frac{\partial F_1}{\partial \omega_i}\, dt \quad , \tag{39}$$

and

$$\dot{y}_i^{(1)} = -\sum_{k=1}^{3} \frac{\partial^2 F_0}{\partial L_i\, \partial L_k}\, x_k^{(1)} - \frac{\partial F_1}{\partial L_i} \quad ,$$

$$y_i^{(1)} = - \sum_{k=1}^{3} \frac{\partial^2 F_0}{\partial L_i \, \partial L_k} \int\int \frac{\partial F_1}{\partial \omega_k} \, dt \, dt - \frac{\partial}{\partial L_i} \int F_1 \, dt \quad . \tag{40}$$

Up to the first order,

$$\nu_i = \nu_i^{(0)} + \epsilon \nu_i^{(1)} = \nu_i^{(0)} \quad .$$

<u>Second Order.</u>

$$\dot{x}_i^{(2)} = \sum_{k=1}^{3} X^{(i)}_{1;\delta_{1k}, \delta_{2k}, \delta_{3k};0,0,0} \, x_k^{(1)} + \sum_{k=1}^{3} X^{(i)}_{1;0,0,0;\delta_{1k}, \delta_{2k}, \delta_{3k}} \, y_k^{(1)}$$

$$+ X_{2;0,0,0;0,0,0} \tag{40}$$

$$\dot{y}_i^{(2)} = - \nu_i^{(2)} + \sum_{k=1}^{3} Z_{\delta_{1k}, \delta_{2k}, \delta_{3k}} \, x_k^{(2)} + \sum_{k=1}^{3} Z^{(i)}_{2\delta_{1k}, 2\delta_{2k}, 2\delta_{3k}} \, {x_k^{(1)}}^2$$

$$+ Z^{(i)}_{1,1,0} \, x_1^{(1)} x_2^{(1)} + Z^{(i)}_{1,0,1} \, x_1^{(1)} x_3^{(1)} + Z^{(i)}_{0,1,1} \, x_2^{(1)} x_3^{(1)}$$

$$- [\dot{x}_i^{(2)}(X \rightarrow Y)] \quad . \tag{41}$$

As was done for the one-dimensional case, we can easily show that $\dot{x}_i^{(2)}$ is free from constant terms; that is, $x_i^{(2)}$ is purely periodic. We can obtain the constant $\nu_i^{(2)}$ by averaging the last five expressions in the right-hand member of $\dot{y}_i^{(2)}$ over ω, from 0 to 2π.

Third Order.

$$\dot{x}_i^{(3)} = \sum_{k=1}^{3} \Big[X^{(i)}_{1;\delta_{1k},\delta_{2k},\delta_{3k};0,0,0}\, x_k^{(2)} + X^{(i)}_{1;0,0,0;\delta_{1k},\delta_{2k},\delta_{3k}}\, y_k^{(2)}$$

$$+ X^{(i)}_{1;2\delta_{1k},2\delta_{2k},2\delta_{3k};0,0,0}\, {x_k^{(1)}}^2 + X^{(i)}_{1;0,0,0;2\delta_{1k},2\delta_{2k},2\delta_{3k}}\, {y_k^{(1)}}^2$$

$$+ X^{(i)}_{1;1,0,0;\delta_{1k},\delta_{2k},\delta_{3k}}\, x_1^{(1)}\, y_k^{(1)} + X^{(i)}_{1;0,1,0;\delta_{1k},\delta_{2k},\delta_{3k}}\, x_2^{(1)}\, y_k^{(1)}$$

$$+ X^{(i)}_{1;0,0,1;\delta_{1k},\delta_{2k},\delta_{3k}}\, x_3^{(1)}\, y_k^{(1)} + X^{(i)}_{2;\delta_{1k},\delta_{2k},\delta_{3k};0,0,0}\, x_k^{(1)}$$

$$+ X^{(i)}_{2;0,0,0;\delta_{1k},\delta_{2k},\delta_{3k}}\, y_k^{(1)} \Big]$$

$$+ X^{(i)}_{1;1,1,0;0,0,0}\, x_1^{(1)}\, x_2^{(1)} + X^{(i)}_{1;1,0,1;0,0,0}\, x_1^{(1)}\, x_3^{(1)}$$

$$+ X^{(i)}_{1;0,1,1;0,0,0}\, x_2^{(1)}\, x_3^{(1)} + X^{(i)}_{1;0,0,0;1,1,0}\, y_1^{(1)}\, y_2^{(1)}$$

$$+ X^{(i)}_{1;0,0,0;1,0,1}\, y_1^{(1)}\, y_3^{(1)} + X^{(i)}_{1;0,0,0;0,1,1}\, y_2^{(1)}\, y_3^{(1)}$$

$$+ X^{(i)}_{3;0,0,0;0,0,0} \ . \qquad (42)$$

$$\dot{y}_i^{(3)} = -\nu_i^{(3)} + \sum_{k=1}^{3} \Big[Z^{(i)}_{\delta_{1k},\delta_{2k},\delta_{3k}}\, x_k^{(3)} + Z^{(i)}_{2\delta_{1k},2\delta_{2k},2\delta_{3k}}\, 2\, x_k^{(1)}\, x_k^{(2)}$$

$$+ Z^{(i)}_{3\delta_{1k},3\delta_{2k},3\delta_{3k}}\, {x_k^{(1)}}^3 \Big] + Z^{(i)}_{1,1,0} \Big[x_1^{(1)}\, x_2^{(2)} + x_1^{(2)}\, x_2^{(1)} \Big]$$

$$+ Z^{(i)}_{1,0,1}\left[x_1^{(1)} x_3^{(2)} + x_1^{(2)} x_3^{(1)}\right] + Z^{(i)}_{0,1,1}\left[x_2^{(1)} x_3^{(2)} + x_2^{(2)} x_3^{(1)}\right]$$

$$+ Z^{(i)}_{2,1,0}\, x_1^{(1)^2} x_2^{(1)} + Z^{(i)}_{1,2,0}\, x_1^{(1)} x_2^{(1)^2} + Z^{(i)}_{2,0,1}\, x_1^{(1)^2} x_3^{(1)}$$

$$+ Z^{(i)}_{1,0,2}\, x_1^{(1)} x_3^{(1)^2} + Z^{(i)}_{0,2,1}\, x_2^{(1)^2} x_3^{(1)} + Z^{(i)}_{0,1,2}\, x_2^{(1)} x_3^{(1)^2}$$

$$+ Z_{1,1,1}\, x_1^{(1)} x_2^{(1)} x_3^{(1)} - [\dot{x}_i^{(3)}(X \rightarrow Y)] \quad . \tag{43}$$

The above relations give a complete third-order solution with all information needed to compute the frequencies ν_i up to a fourth-order accuracy in the small parameter. From simple inspection we can write the differential equations for $\dot{x}_i^{(n)}$, $\dot{y}_i^{(n)}$ for any value of n. But the important point is that such equations as (36) and (37) can easily be developed by an electronic computer, either in algebraic form or in numerical form. In fact, since the solution of order n - 1 gives the frequency up to order n, the right-hand members of Eqs. (36) and (37) can be harmonically analyzed by numerical means to give the solution correct to the nth order in the form of Fourier series with numerical coefficients. This, of course, implies a knowledge of the constants $\underset{\sim}{L}^0$, $\underset{\sim}{\beta}$. In the next section we discuss some of these matters.

4. SEMINUMERICAL APPROACH

In several problems of orbit determination, for example, those involving artificial satellites, it is not possible nor practical to develop the disturbing function analytically in terms of the mean anomaly of the disturbed body, since this would require series expansions in the eccentricity and the sine of the inclination, which might not be small with respect to unity. This, however, does not mean that a numerical Fourier analysis of the disturbing function cannot be obtained. Such trigonometric series with numerical coefficients can usually be computed, with little effort, to any desired degree of precision. In order to apply the theory described in this paper to problems involving high eccentricity and inclination, we must consider $F(\underset{\sim}{L}^0, \underset{\sim}{\omega})$ given by a trigonometric series, with numerical coefficients. The functions $F_p(\underset{\sim}{L}^0, \underset{\sim}{\omega})$ are then selected by inspection of those coefficients. For particular values of L_1^0, L_2^0, L_3^0, β_1, β_2, β_3, which define the initial conditions, the solution requires a process of differential orbit improvement.

The series solutions given by this method can be represented as

$$\ell_i = \nu_i(L^0;\epsilon)t + \beta_i + \sum_{p=1}^{N} \epsilon^p \sum_{|\underset{\sim}{k}| \leq M_p} \Big[A_{p;\underset{\sim}{k}}^{(i)}(L^0) \cos(\underset{\sim}{k} \cdot \underset{\sim}{\omega}) + B_{p;\underset{\sim}{k}}^{(i)}(L^0) \sin(\underset{\sim}{k} \cdot \underset{\sim}{\omega}) \Big] \tag{44}$$

$$L_i = L_i^0 + \sum_{p=1}^{N} \epsilon^p \sum_{|\underset{\sim}{k}| \leq M_p} \Big[C_{p;\underset{\sim}{k}}^{(i)}(L^0) \cos(\underset{\sim}{k} \cdot \underset{\sim}{\omega}) + D_{p;\underset{\sim}{k}}^{(i)}(L^0) \sin \underset{\sim}{k} \cdot \underset{\sim}{\omega} \Big] , \tag{45}$$

where $\omega_i = \nu_i t + \beta_i$, and N is related to the precision achieved by iteration. At $t = 0$, Eqs. (44) and (45) reduce to

$$\ell_i(0) = \beta_i + \sum_{p=1}^{N} \epsilon^p \sum_{|\underset{\sim}{k}| \leq M_p} \Big[A_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^0) \cos(\underset{\sim}{k} \cdot \underset{\sim}{\beta}) + B_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^0) \sin(\underset{\sim}{k} \cdot \underset{\sim}{\beta}) \Big] \tag{46}$$

$$L_i(0) = L_i^0 + \sum_{p=1}^{N} \epsilon^p \sum_{|\underset{\sim}{k}| \le M_p} \left[C_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^0) \cos(\underset{\sim}{k} \cdot \underset{\sim}{\beta}) + D_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^0) \sin(\underset{\sim}{k} \cdot \underset{\sim}{\beta}) \right] , \tag{47}$$

and for $\epsilon = 0$, $\beta_i^0 = \ell_i(0)$, $L_i^{00} = L_i(0)$, which gives a zero-order solution to β_i, L_i^0 in terms of the initial conditions. The $(\underline{n}+1)$th order is obtained by recurrence from the inversion of Eqs. (46) and (47); that is,

$$\beta_i^{(n+1)} = \ell_i(0) - \sum_{p=1}^{N} \epsilon^p \sum_{|\underset{\sim}{k}| \le M_p} \left[A_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^{0n}) \cos(\underset{\sim}{k} \cdot \underset{\sim}{\beta}^n) + B_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^{0n}) \sin(\underset{\sim}{k} \cdot \underset{\sim}{\beta}^n) \right] \tag{48}$$

$$L_i^{0n} = L_i(0) - \sum_{p=1}^{N} \epsilon^p \sum_{|\underset{\sim}{k}| \le M_p} \left[C_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^{0n}) \cos(\underset{\sim}{k} \cdot \underset{\sim}{\beta}^n) + D_{p;\underset{\sim}{k}}^{(i)}(\underset{\sim}{L}^{0n}) \sin(\underset{\sim}{k} \cdot \underset{\sim}{\beta}^n) \right] . \tag{49}$$

Nevertheless, this process of iteration requires that we know the series solutions (44) and (45), which in turn can only be obtained after the numerical values of $\underset{\sim}{L}^0$, $\underset{\sim}{\beta}$ are introduced. It is at this stage that a differential correction scheme untangles the situation, assuming, of course, ϵ to be reasonably small. Thus, given the initial conditions $[\underset{\sim}{L}(0), \ell(0)]$, we can assume the approximate relation $L_i^0 = L_i(0)$, $\beta_i = \ell_i(0)$ and construct an approximate orbit given by Eqs. (44) and (45). Once these series are known, better values for L_i^0 and β_i can be found by one iteration of Eqs. (48) and (49). A new orbit is then obtained in the form given again by Eqs. (44) and (45). The iteration of such a process for small ϵ will usually converge to the desired accuracy.

Suppose, therefore, we have provisional or definitive values of L_i^0 and β_i. The functions $\partial F_0/\partial L_i$, $\partial F_p/\partial L_i$, $\partial F_p/\partial \ell_i$ ($p = 1, 2, 3, \ldots, N$) and their partial derivatives up to order N can then be reduced to Fourier series with numerical coefficients. Let

$$-\frac{1}{n!}\binom{n}{\alpha}\binom{n-\alpha}{\beta}\frac{\partial^n}{\partial L_1^{n-\alpha-\beta}\,\partial L_2^{\beta}\,\partial L_3^{\alpha}}\left(\frac{\partial F_0}{\partial L_i^0}\right) = M^{(i)}_{n-\alpha-\beta,\beta,\alpha} \quad ,$$

$$\frac{1}{n!}\binom{n}{\alpha}\binom{n-\alpha}{\beta}\binom{n-\alpha-\beta}{\gamma}\binom{\alpha}{\delta}\binom{\alpha-\delta}{\mu}\frac{\partial^n}{\partial L_1^{n-\alpha-\beta-\gamma}\,\partial L_2^{\gamma}\,\partial L_3^{\beta}\,\partial\omega_1^{\alpha-\delta-\mu}\,\partial\omega_2^{\mu}\,\partial\omega_3^{\delta}}\left[\frac{\partial F_p(\underset{\sim}{L}^0,\underset{\sim}{\omega})}{\partial\omega_i}\right]$$

$$= \sum_{\underset{\sim}{k}}\Big(A^{(i)}_{p;\underset{\sim}{k};n-\alpha-\beta-\gamma,\gamma,\beta;\alpha-\delta-\mu,\delta,\mu}\cos\underset{\sim}{k}\cdot\underset{\sim}{\omega}$$

$$+ B^{(i)}_{p;\underset{\sim}{k};n-\alpha-\beta-\gamma,\alpha,\beta;\alpha-\delta-\mu,\delta,\mu}\sin\underset{\sim}{k}\cdot\underset{\sim}{\omega}\Big) \quad ,$$

and

$$\frac{1}{n!}\binom{n}{\alpha}\binom{n-\alpha}{\beta}\binom{n-\alpha-\beta}{\gamma}\binom{\alpha}{\delta}\binom{\alpha-\delta}{\mu}\frac{\partial^n}{\partial L_1^{n-\alpha-\beta-\gamma}\,\partial L_2^{\gamma}\,\partial L_3^{\beta}\,\partial\omega_1^{\alpha-\delta-\mu}\,\partial\omega_2^{\mu}\,\partial\omega_3^{\delta}}\left[\frac{\partial F_p(\underset{\sim}{L}^0,\underset{\sim}{\omega})}{\partial L_i}\right]$$

$$= \sum_{\underset{\sim}{k}}\Big(C^{(i)}_{p;\underset{\sim}{k};n-\alpha-\beta-\gamma,\alpha,\beta;\alpha-\delta-\mu,\delta,\mu}\cos\underset{\sim}{k}\cdot\underset{\sim}{\omega}$$

$$+ D^{(i)}_{p;\underset{\sim}{k};n-\alpha-\beta-\gamma,\alpha,\beta;\alpha-\delta-\mu,\delta,\mu}\sin\underset{\sim}{k}\cdot\underset{\sim}{\omega}\Big) \quad .$$

The numbers A, B, C, D are readily related to the coefficients of the series X, Y defined by relations (30) and (32), and the numbers M give the values of the functions Z. In fact,

$$M^{(i)}_{n-\alpha-\beta,\beta,\alpha} = Z^{(i)}_{n-\alpha-\beta,\beta,\alpha} \quad ,$$

$$\sum_{\boldsymbol{k}} \left(A^{(i)}_{p;\boldsymbol{k};n-\alpha-\beta-\gamma,\gamma,\beta;a-\delta-\mu,\delta,\mu} \cos \boldsymbol{k} \cdot \boldsymbol{\omega} \right.$$

$$\left. + B^{(i)}_{p;\boldsymbol{k};n-\alpha-\beta-\gamma,\gamma,\beta;\alpha-\delta-\mu,\delta,\mu} \sin \boldsymbol{k} \cdot \boldsymbol{\omega} \right)$$

$$= X^{(i)}_{p;n-\alpha-\beta-\gamma,\gamma,\beta;\alpha-\delta-\mu,\delta,\mu}$$

and

$$\sum_{\boldsymbol{k}} \left(C^{(i)}_{p;\boldsymbol{k};n-\alpha-\beta-\gamma,\gamma,\beta;\alpha-\delta-\mu,\delta,\mu} \cos \boldsymbol{k} \cdot \boldsymbol{\omega} \right.$$

$$\left. + D^{(i)}_{p;\boldsymbol{k};n-\alpha-\beta-\gamma,\gamma,\beta;\alpha-\delta-\mu,\delta,\mu} \sin \boldsymbol{k} \cdot \boldsymbol{\omega} \right)$$

$$= Y^{(i)}_{p;n-\alpha-\beta-\gamma,\gamma,\beta;\alpha-\delta-\mu,\delta,\mu} \quad .$$

In the above relations we must have

$$n = 0, 1, 2, \ldots, N$$

$$\alpha = 0, 1, 2, \ldots, n$$

$$\beta = 0, 1, 2, \ldots, n - \alpha$$

$$\gamma = 0, 1, 2, \ldots, n - \alpha - \beta$$

$$\delta = 0, 1, 2, \ldots, \alpha$$

$$\mu = 0, 1, 2, \ldots, \alpha - \delta$$

and the maximum value of $|\boldsymbol{k}|$, M_p, must be decided according to the precision sought.

It is worthwhile to note that the iteration from the initial conditions to the constants of integration is also needed in the classical theories generated by von Zeipel's method. The essential differences are that we are able here to write rapidly the equations for any order of approximation and that we can perform a numerical harmonic analysis of the right-hand members of the differential equations of motion, however complicated they might be, using as many points as necessary to reach a prescribed precision in the final solution.

We have made no distinction between short-periodic, long-periodic, and secular perturbations, which are treated globally by the method described. However, such a distinction is always possible by simple elimination of the terms not needed in the Fourier series.

In conclusion, we emphasize that if one is ready to elaborate long trigonometric series, either by hand or by computer, either numerically or analytically, the theory we have described can be extended in a simple and systematic way to any order of approximation.

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